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The Ahlfors-Shimizu Characteristic Function and the Cartan Identity on Riemann Surfaces

by

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§1. Introduction

We shall consider a counterpart of the Cartan identity in terms of the Ahlfors-Shimizu characteristic function. Although no analogous identity is obtained, the "almost" Cartain identity in our main theorem might be of interest; the result is sharp.

Each point of a Riemann surface S will be identified, as usual, with its local-parametric image in the complex plane $C=\{|z|<\infty\}$. We fix a point $P\in S$ once and for all, and let $\mathscr{D}$ be the family of subdomains D of S such that (i) D contains P, (ii) the closure $\partial D\cup D$ is compact, and (iii) the boundary ∂D consists of a finite number of mutually disjoint, analytic, simple and closed curves. The radius $r(D)>0$ of $D\in\mathscr{D}$ is defined by

$$\log r(D)=\lim_{z\to P}\{g_D(z, P)+\log|z-P|\},$$

where $g_D(z, P)$ is the Green function of D with its pole at P. Set

$$D_t=\{z\in D;\ g_D(z, P)\geqq\log(r/t)\}, \qquad 0<t<r=r(D).$$

Let $M(S)$ be the family of meromorphic functions on S. The spherical derivative $f^{\#}=|f'|/(1+|f|^2)$ of $f\in M(S)$ is, in general, not a function on S, yet the second order differential $f^{\#}(z)^2dxdy$ $(z=x+iy)$ is well defined on S. The Ahlfors-Shimizu characteristic function of $f\in M(S)$ is then the function

$$T_A(D,f)=\pi^{-1}\int_0^{r(D)} t^{-1}\left\{\iint_{D_t} f^{\#}(z)^2dxdy\right\}dt, \qquad D\in\mathscr{D}.$$

One should note that $r(D)$ depends on a choice of a local parameter at P, yet $T_A(D,f)$ does not depend on it. It is not difficult to observe that

$$T_A(D,f)=\pi^{-1}\iint_D f^{\#}(z)^2 g_D(z, P)dxdy;$$

see the forthcoming paper [3] for the details.

However, to avoid confusion, we shall let $r(D)$ definite by taking the limit as $z \to P$ in the local parametric disk with center P.

The terminology "Ahlfors-Shimizu characteristic function" is justified because if $S=\{|z|<R\leqq+\infty\}$, $P=0$, and $D=\{|z|<r<R\}$, we obtain the familiar

$$\pi^{-1}\int_0^r t^{-1}\left\{\iint_{|z|<t} f^{\#}(z)^2 dxdy\right\}dt .$$

Returning to our general situation, we let $C^*=C\cup\{\infty\}$. For each $a\in C^*$ we let $n(t,a,f)$ be the number of the roots of the equation $f(z)=a$ in D_t, the multiplicities being counted. Letting

$$n(0,a,f)=\lim_{t\to 0} n(t,a,f) ,$$

we set

$$N(D,a,f)=\int_0^{r(D)}\frac{n(t,a,f)-n(0,a,f)}{t}dt+n(0,a,f)\log r(D) .$$

Furthermore, we set

$$m(D,f)=-(2\pi)^{-1}\int_{\partial D}\log^+|f(z)|\,dg_D^*(z,P) , \quad \text{and}$$

$$m_A(D,f)=-(4\pi)^{-1}\int_{\partial D}\log(1+|f(z)|^2)dg_D^*(z,P) ,$$

where $dg_D^*(z,P)$ is the conjugate differential of $dg_D(z,P)$ and the integrations are in the positive sense.

For the Nevanlinna characteristic function

$$T(D,f)=m(D,f)+N(D,\infty,f) , \qquad D\in\mathscr{D} ,$$

the celebrated Cartan identity reads that if $f(P)\neq\infty$, then

$$T(D,f)=\log^+|f(P)|+(2\pi)^{-1}\int_0^{2\pi}N(D,e^{it},f)dt ; \tag{1.1}$$

see, for example, ([2], (56), p. 89), an obvious extension of the classicial one ([1], (3.9), p. 177).

The chordal distance on C^* as the Riemann sphere is

$$k(z,w)=\frac{|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}} , \qquad z,w\in C^* ,$$

with the usual device for ∞. Set for $a\in C^*$, and for ρ, $0<\rho<1$,

$$\gamma(a,\rho)=\{z\in C^*; k(z,a)=\rho\} .$$

The circle $\gamma(a,\rho)$ has the length

$$l(\rho)=2\pi\rho(1-\rho^2)^{1/2}\,.$$

Letting $dk(w)=(1+|w|^2)^{-1}|dw|$ for $w\in \boldsymbol{C}^*$ we consider the mean

$$I(D,a,\rho,f)=l(\rho)^{-1}\int_{\gamma(a,\rho)}N(D,w,f)dk(w)\,.$$

Then,

$$I(D,a,\rho,f)=\int_0^{r(D)}t^{-1}\left\{l(\rho)^{-1}\int_{\gamma(a,\rho)}n(t,w,f)dk(w)\right\}dt$$

and

$$\int_{\gamma(a,\rho)}n(t,w,f)dk(w)$$

is the total length of the curves on the Riemannian image of D_t by f lying over $\gamma(a,\rho)$.

THEOREM. *For each quartet, $f\in M(S)$, $a\in \boldsymbol{C}^*$, $0<\rho<1$, and $D\in\mathscr{D}$, the following estimate always holds.*

$$|T_A(D,f)-I(D,a,\rho,f)|\leqq -\log\{\min(\rho,(1-\rho^2)^{1/2})\}\,. \tag{1.2}$$

For the antipodal point $a^*=-1/\bar{a}$ of $a\in \boldsymbol{C}^*$ we have $\gamma(a,\rho)=\gamma(a^*,(1-\rho^2)^{1/2})$. The minimum in the right-hand side of (1.2) is the smaller of the radii, ρ, $(1-\rho^2)^{1/2}$.

The right-hand side of (1.2), as a function of ρ, $0<\rho<1$, attains its minimum $\log\sqrt{2}$ at $\rho=1/\sqrt{2}$. In this case the circle $\gamma(a,1/\sqrt{2})$ is a great circle on $\boldsymbol{C}^*$; in particular, $\gamma(0,1/\sqrt{2})$ is the equator.

§ 2. The sharpness

We shall prove that (1.2) is sharp for $S=\boldsymbol{C}$ and $P=0$. Set $f(z)=z$. Then, given ρ, $0<\rho<1$, we may find $a\in \boldsymbol{C}^*$ and $r>0$ such that the equality in (1.2) holds for $D=\varDelta(r)$, where $\varDelta(t)=\{|z|<t\}$, $t>0$.

First of all,

$$T_A(\varDelta(r),f)=(1/2)\log(1+r^2)\,,\qquad r>0\,. \tag{2.1}$$

Set

$$\phi(t)=t(1-t^2)^{-1/2}\,,\qquad 0<t<1\,.$$

Since

$$N(\varDelta(r),w,f)=\log^+|r/w|\,,\qquad w\in \boldsymbol{C}^*\,,$$

it follows that, for $r>0$ and $0<t<1$,

(2.2)
$$I(\Delta(r), 0, t, f)=I(\Delta(r), \infty, (1-t^2)^{1/2}, f)$$
$$=l(t)^{-1}\int_{|w|=\phi(t)}\log^+|r/w|\,dk(w)=\log^+(r/\phi(t)).$$

For $0<\rho\leqq 1/\sqrt{2}$ we let $a=\infty$ and $r=1/\phi(\rho)$. Then, (2.1) and (2.2) for $t=(1-\rho^2)^{1/2}$ show that the left-hand side of (1.2) is precisely $-\log\rho$ which is equal to the right-hand side of (1.2).

For $1/\sqrt{2}<\rho<1$ we let $a=0$ and $r=\phi(\rho)$. By the same reasoning, with $t=\rho$, we have the equality in (1.2).

§3. Proof of the Theorem

Suppose that $F\in M(S)$ and $|F(P)|\leqq c$, $0<c<+\infty$. We shall show a key inequality

(3.1)
$$|T(D, F/c)-T_A(D, F)|\leqq(1/2)\log\{\max(1+c^2, 1+c^{-2})\}.$$

First of all,

$$T_A(D, F)=\pi^{-1}\iint_D F^\#(z)^2 g_D(z, P)dxdy.$$

It then follows from the Green formula with

$$\{\Delta\log(1+|F(z)|^2)\}dxdy=4F^\#(z)^2 dxdy,$$

that

$$T_A(D, F)=m_A(D, F)+N(D, \infty, F)-(1/2)\log(1+|F(P)|^2).$$

Since $N(D, \infty, F)=N(D, \infty, F/c)$, it follows then that

(3.2)
$$T(D, F/c)-T_A(D, F)$$
$$=m(D, F/c)-m_A(D, F)+(1/2)\log(1+|F(P)|^2)$$
$$=-(2\pi)^{-1}\int_{\partial D}\Phi(|F(z)|)dg_D^*(z, P)+(1/2)\log(1+|F(P)|^2),$$

where

$$\Phi(x)=\log^+(x/c)-(1/2)\log(1+x^2), \qquad 0\leqq x\leqq+\infty.$$

Elementary analysis on Φ shows that

$$-(1/2)\log(1+c^2)\leqq\Phi(x)\leqq 0 \qquad \text{for } 0\leqq x\leqq c;$$
$$-(1/2)\log(1+c^2)\leqq\Phi(x)\leqq-\log c \qquad \text{for } c\leqq x\leqq+\infty.$$

Since $0\leqq(1/2)\log(1+|F(P)|^2)\leqq(1/2)\log(1+c^2)$, we have (3.1) from

$$-(1/2)\log(1+c^2)\leqq T(D, F/c)-T_A(D, F)$$
$$\leqq(1/2)\log(1+c^2)+\max(0, -\log c)\,.$$

Fix a, ρ, and D. For

$$H(w)=(w-a)/(1+\bar{a}w)\,, \qquad \text{if} \quad a\neq\infty\,,$$
$$=1/w\,, \qquad \text{if} \quad a=\infty\,,$$

we consider the composed function $h=H\circ f$. Then,

$$T_A(D,f)=T_A(D, h)=T_A(D, 1/h)\,.$$

Since $dk(v)=dk(w)$ for $v=H(w)$, we have

$$dk(w)=(1-\rho^2)\,|\,dv\,| \qquad \text{for} \quad w\in\gamma(a, \rho)\,,$$

so that

$$I(D, a, \rho, f)=l(\rho)^{-1}(1-\rho^2)\int_{|v|=\phi(\rho)} N(D, v, h)\,|\,dv\,|\,.$$

Therefore,

$$I(D, a, \rho, f)=(2\pi)^{-1}\int_0^{2\pi} N(D, e^{it}, h/\phi(\rho))dt\,, \tag{3.3}$$

and the change of the variable: $s=2\pi-t$ yields

$$I(D, a, \rho, f)=(2\pi)^{-1}\int_0^{2\pi} N(D, e^{is}, \phi(\rho)/h)ds\,. \tag{3.4}$$

In the case $|\,h(P)/\phi(\rho)\,|\leqq 1$, the identity (1.1) for $h/\phi(\rho)$, together with (3.3) yields that

$$I(D, a, \rho, f)=T(D, h/\phi(\rho))\,.$$

Therefore, it follows from (3.1) for $F=h$ and $c=\phi(\rho)$, that

$$|\,I(D, a, \rho, f)-T_A(D,f)\,|=|\,T(D, h/\phi(\rho))-T_A(D, h)\,|$$
$$\leqq(1/2)\log\{\max((1-\rho^2)^{-1}, \rho^{-2})\}\,,$$

whence (1.2).

In the case $|\,h(P)/\phi(\rho)\,|\geqq 1$, the identity (1.1) for $\phi(\rho)/h$ with (3.4) yields that

$$I(D, a, \rho, f)=T(D, \phi(\rho)/h)\,.$$

Therefore, it follows from (3.1) for $F=1/h$ and $c=1/\phi(\rho)$, that

$$|\,I(D, a, \rho, f)-T_A(D,f)\,|=|\,T(D, \phi(\rho)/h)-T_A(D, 1/h)\,|$$
$$\leqq(1/2)\log\{\max(\rho^{-2}, (1-\rho^2)^{-1})\}\,,$$

whence (1.2).

§ 4. Concluding remarks

Let f and h be as in the proof of the Theorem. We assume first that $0<\alpha\equiv |h(P)|<\infty$. It then follows from (3.3), (3.4) and (1.1) that

$$I(D, a, \rho, f)=T(D, h/\phi(\rho)) \quad \text{if} \quad \alpha/\phi(\rho)\leqq 1\,, \tag{4.1}$$

and that

$$I(D, a, \rho, f)=T(D, \phi(\rho)/h) \quad \text{if} \quad \alpha/\phi(\rho)\geqq 1\,. \tag{4.2}$$

In case (4.1) we deduce from (3.2) that

$$\begin{aligned} I(D, a, \rho, f)&=T_A(D, h)+(1/2)\log(1+\alpha^2)\\ &\quad+(2\pi)^{-1}\int_{\partial D}\{(1/2)\log(1+|h(z)|^2)-\log^+|h(z)/\phi(\rho)|\}dg_D^*(z, P)\end{aligned}$$

and in case (4.2) that

$$\begin{aligned} I(D, a, \rho, f)&=T_A(D, 1/h)+(1/2)\log\{1+(1/\alpha)^2\}\\ &\quad+(2\pi)^{-1}\int_{\partial D}\{(1/2)\log(1+|1/h(z)|^2)-\log^+|\phi(\rho)/h(z)|\}dg_D^*(z, P)\,.\end{aligned}$$

Combining these estimates with the fact that $T_A(D, f)=T_A(D, h)=T_A(D, 1/h)$, we obtain

$$\begin{aligned} &\int_0^1 I(D, a, \rho, f)d\rho-T_A(D, f)\\ &\quad=-(2\pi)^{-1}\int_{\partial D}(J_1(|h(z)|)+J_2(|h(z)|))dg_D^*(z, P)\,,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned} J_1(x)&=\int_{\phi^{-1}(\alpha)}^1 (1/2)\{\log(1+\alpha^2)-\log(1+x^2)\}d\rho\\ &\quad+\int_0^{\phi^{-1}(\alpha)}(1/2)[\log\{1+(1/\alpha)^2\}-\log\{1+(1/x)^2\}]d\rho\\ &=(1/2)\log(1+\alpha^2)+\phi^{-1}(\alpha)\log(x/\alpha)-(1/2)\log(1+x^2)\end{aligned}$$

and

$$\begin{aligned}J_2(x)&=\int_{\phi^{-1}(\alpha)}^{1}\log^+(x/\phi(\rho))d\rho+\int_0^{\phi^{-1}(\alpha)}\log^+(\phi(\rho)/x)d\rho\\&=\int_{\phi^{-1}(\alpha)}^{\phi^{-1}(x)}\log(x/\phi(\rho))d\rho=\int_\alpha^x\log(x/t)d(t(1+t^2)^{-1/2})\\&=-\alpha(1+\alpha^2)^{-1/2}\log(x/\alpha)+\log\{x+(1+x^2)^{1/2}\}\\&\quad-\log\{\alpha+(1+\alpha^2)^{1/2}\}.\end{aligned}$$

These estimates, together with $\phi^{-1}(\alpha)=\alpha(1+\alpha^2)^{-1/2}$, imply that, for $0<x<\infty$,

$$J_1(x)+J_2(x)=\log\{1+x(1+x^2)^{-1/2}\}-\log\{1+\alpha(1+\alpha^2)^{-1/2}\},$$

which, together with (4.3), further implies that,

$$\left|\int_0^1 I(D,a,\rho,f)d\rho-T_A(D,f)\right|\leqq\log 2 \tag{4.4}$$

in case $0<\alpha<\infty$. It is easy to observe that (4.4) holds also for $\alpha=0$ and $\alpha=\infty$.

For $f(z)=z$, considered in §2, the formulae (2.1) and (2.2) show that

$$\begin{aligned}\int_0^1 I(\Delta(r),0,\rho,f)d\rho-T_A(\Delta(r),f)&=\int_0^1\log^+(r/\phi(\rho))d\rho-(1/2)\log(1+r^2)\\&=\log\{1+r(1+r^2)^{-1/2}\}\longrightarrow\log 2\qquad\text{as}\quad r\longrightarrow\infty;\end{aligned}$$

thus, (4.4) is sharp.

Let f and h be as in the proof of the Theorem. Setting $r=\phi(\rho)$, $0<\rho<1$, in the identity:

$$T_A(D,h)=\pi^{-1}\int_0^\infty\int_0^{2\pi}N(D,re^{it},h)(1+r^2)^{-2}rdrdt$$

we obtain

$$T_A(D,f)=T_A(D,h)=\int_0^1\left\{\pi^{-1}\int_0^{2\pi}N(D,e^{it},h/\phi(\rho))dt\right\}\rho d\rho.$$

This, together with (3.3), shows that

$$T_A(D,f)=2\int_0^1 I(D,a,\rho,f)\rho d\rho \tag{4.5}$$

for each trio, $f\in M(S)$, $a\in \boldsymbol{C}^*$, and $D\in\mathscr{D}$.

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